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Reduced Order Modeling of a Bladed Rotor with Geometric Mistuning: Alternative Bases and Extremely Large Mistuning

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ABSTRACT

This paper represents further development of Modified Modal Domain Analysis (MMDA) (Sinha, 2009), which is a breakthrough method for the reduced-order modeling of a bladed rotor with geometric mistuning. The bases vectors for model reduction in MMDA have been formed using the mode shapes of cyclic sectors with blades' geometries perturbed along the POD (Proper Orthogonal Decomposition) features. The use of mode shapes from modal analyses of cyclic sectors perturbed along the POD features adds an additional step of creating the finite element models of artificially perturbed geometries. Here, an alternative formulation of MMDA is presented in which bases vectors are created from cyclic sectors with actual blades. Therefore, the additional step of creating artificial blades with geometries perturbed along POD features is avoided.

The MMDA is also extended to a bladed rotor in which a few blades have extremely large mistuning; for example, blended airfoils. The validity of proposed approaches is shown by comparing with ANSYS results for full (360 degree) bladed rotor.

INTRODUCTION

Geometric mistuning (Bhartiya and Sinha, 2011a, b, Brown, 2008, Ganine et al., 2009, Lim et al., 2004, Sinha 2009 and Sinha and Bhartiya, 2010) refers to inevitable variations in blades' geometries, which arise during the manufacturing process. The distinctive feature of geometric mistuning is that it leads to simultaneous and dependent perturbations in mass and stiffness matrices of each blade. It has been an active area of research primarily because it can lead to significant increase in the maximum vibratory stress in the bladed rotor. Other important aspects of mistuning are that the cyclic symmetry is lost and the mistuning parameters are random variables. Both these aspects increase the computational requirements to assess the impact of mistuning. Because of the loss of cyclic symmetry, sector analysis cannot be performed in a standard finite element package such as ANSYS or NASTRAN. And because of the stochastic aspect of mistuning, Monte Carlo simulation will be required to compute the statistics of the maximum amplitude of blade vibration.

In order to analyze the impact of geometric mistuning, a breakthrough reduced-order modeling approach has been provided by Sinha (2009), which has been called Modified Modal Domain Approach (MMDA). This approach is based on sector analyses and POD analysis (Sinha et al., 2008) of geometry variations and has been shown to provide exact mode shapes and natural frequencies for a bladed rotor with geometric mistuning. The detailed algorithm for MMDA is presented in papers by Sinha (2009) and Sinha and Bhartiya (2010). Here, the algorithm is summarized. Let the dynamics of a mistuned bladed rotor be given by

$$M\ddot{\mathbf{x}} + K\mathbf{x} = 0 \tag{1}$$

where M and K are mass and stiffness matrices of the full (360 degrees) rotor, respectively. The model reduction is achieved by the following transformation:

$$\mathbf{x} = \Phi\mathbf{z} \tag{2}$$

where

$$\Phi = [\Phi_0 \quad \Phi_1 \quad \cdots \quad \Phi_{np}] \tag{3}$$

and Φ_i is composed of mode shapes of a tuned disk with blades' geometries perturbed along i^{th} POD feature and np is the number of POD features (Sinha et al., 2008). From (1) and (2),

$$M_r\ddot{\mathbf{z}} + K_r\mathbf{z} = 0 \tag{4}$$

where

$$M_r = \Phi^H M_t\Phi + \Phi^H \delta M\Phi \tag{5}$$

and

$$K_r = \Phi^H K_t\Phi + \Phi^H \delta K\Phi \tag{6}$$

M_t & K_t are the mass and stiffness matrices of the full (360 degrees) tuned system with average geometry and δM & δK are perturbations in mass and stiffness matrices due to mistuning. The details for calculation of $\Phi^H M_t\Phi$ & $\Phi^H K_t\Phi$ and $\Phi^H \delta M\Phi$ & $\Phi^H \delta K\Phi$ from sector analyses are given in the papers by Sinha (2009), and Sinha and Bhartiya (2010). Bhartiya and Sinha (2011b) have also estimated δM & δK via Taylor series expansions in terms of POD features and have presented an efficient algorithm for Monte Carlo simulation.

In the earlier papers ((Sinha, 2009), (Sinha and Bhartiya, 2010) and (Bhartiya and Sinha, 2011a, b)), bases for MMDA have been formed using the mode shapes of cyclic sectors perturbed along the POD features. The use of mode shapes from modal analysis of cyclic sectors perturbed along the POD features adds an additional step of creating the finite element models of artificially perturbed geometries. The first part of the paper presents an alternative

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formulation of MMDA which avoids the use of mode shapes from geometries perturbed along POD features to form the bases.
Another problem that arises in mistuning is that of the blades with extremely large geometric mistuning (rogue blade) caused by foreign object damage (FOD) or blade tip blending. In such cases the techniques developed for mistuning ((Sinha, 2009), (Sinha and Bhartiya, 2010) and (Bhartiya and Sinha, 2011a, b)) with a conventional statistical distribution may fail to provide accurate results and suitable modifications in the algorithm are required to obtain accurate reduced order model for extremely large mistuning. The second part of the paper presents a reduced order model for extremely large mistuning based on MMDA.

ALTERNATIVE BASES

In MMDA the true mode shapes of a mistuned bladed disk assembly are approximated by a linear combination of mode shapes of 'average' tuned geometry and tuned geometries of sectors with blades perturbed along the POD features as given by Eq. (3). The modes for the geometries from the POD analysis are used because POD analysis provides independent vectors for perturbations in geometries, and by using only dominant POD features to form the bases of geometric perturbations; a minimal set of mode shapes is obtained to form the solution bases. The idea behind POD analysis is to obtain independent perturbation vectors and since the perturbation in actual sector can be represented as a linear combination of the POD vectors using KL expansion ((Sinha et al., 2008) and (Ghanem and Spanos, 1991)), it is proposed that alternatively the actual mistuned sectors may themselves be used to form suitable bases; i.e.,

$$\mathbf{x} = \Phi \mathbf{z} \tag{7}$$

where

$$\Phi = [\Phi_0 \quad \Phi_{a1} \quad \Phi_{a2} \quad . \quad . \quad \Phi_{an}] \tag{8}$$

Eq. (8) is similar to Eq. (3) but the mode shapes from actual sectors are used to form the bases. Here Φ_0 are mode shapes obtained from the modal analysis of the 'average' tuned bladed disk assembly and Φ_{al} are the mode shapes from the modal analysis of the tuned bladed disk assembly with each sector represented by the sector #*l* of the actual mistuned bladed disk. Since only a few POD features are dominant, the sectors with geometries most aligned along the dominant POD features can be used to form bases. Hence using the alternative bases, the steps for the MMDA analysis can be enumerated as follows:

1. Identify independent geometric perturbation vectors using POD analysis of Coordinate Measurement Machine (CMM) data for mistuned blades.
2. Identify blades/sectors most aligned with the dominant POD features.
3. Use mode shapes from cyclic analyses of identified sectors to form the bases of solution.
4. Generate and solve reduced order model using alternative bases.

In the next section MMDA analysis is run with alternative bases and the results are presented.

Numerical Results

In this section MMDA analysis is run for the academic rotor considered by Sinha (2009), and the finite element model is given by Figure 1. The sector has 24 blades, with 1980 independent and unconstrained degrees of freedom per sector.

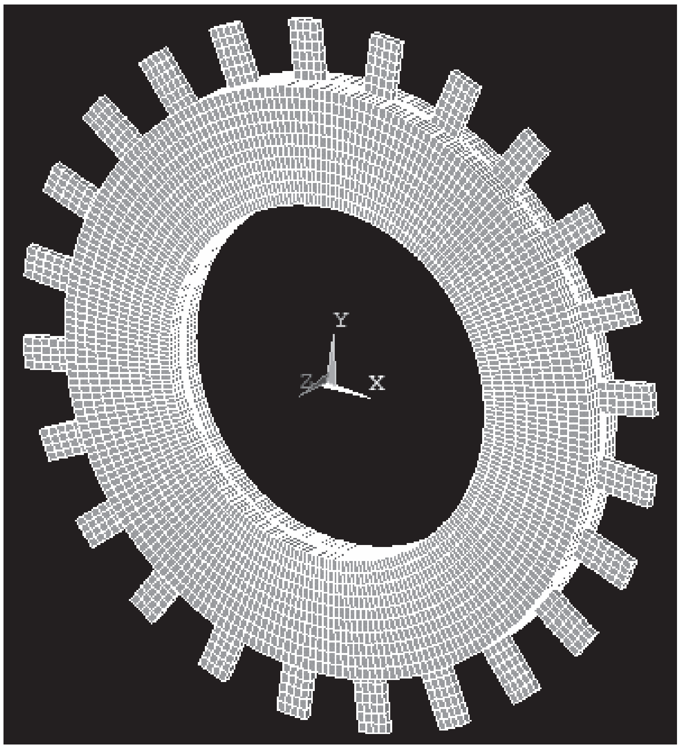

Figure 1: Finite element model of the academic rotor (Sinha, 2009)

The dominant mistuning is again introduced along the two POD features (#1 and #2 in Fig. 2) as discussed by Sinha (2009), i.e. by scaling the thicknesses of the blades with average geometry by a factor (Eq. 9) and rotating the surface of the blade by an angle of θ_l from the vertical (Eq. 10):

$$b_l = b_0(1 + \xi_{1l}) \tag{9}$$

$$\theta_l = \tan^{-1}(\frac{b_0 \xi_{2l}}{L}) \tag{10}$$

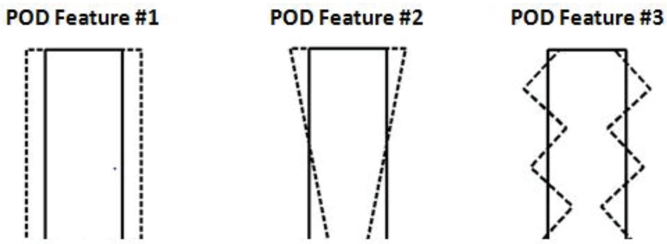

Figure 2: POD features in mistuned bladed disk assembly

The mistuning parameter values (ξ_1 and ξ_2) for POD #1 and #2 are given in Figures 3 and 4 respectively. The means of mistuning parameter values are almost zero and the standard deviations are 0.017 and 0.015 respectively. Note that mean (μ) is not exactly zero because of a finite number of random variables. The maximum value of deviation in the blade thickness is 3% of the average blade thickness.

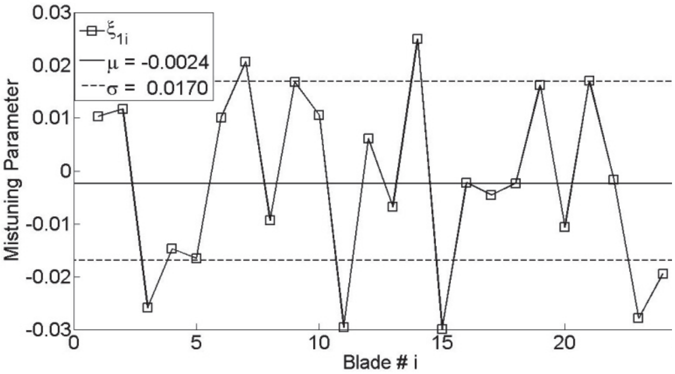

Figure 3: Mistuning pattern for POD #1

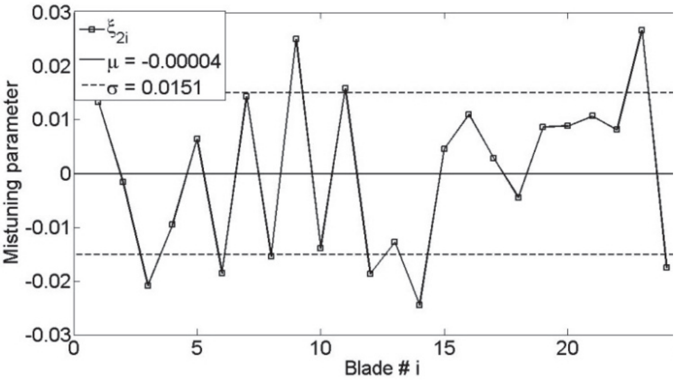

Figure 4: Mistuning pattern for POD #2

A look at the values of mistuning parameters for each blade shows that for sectors 12 (ξ_1 = 0.0061302, ξ_2 = -0.0186800) and 15 (ξ_1 = -0.029931, ξ_2 = 0.0045726), mistuning parameters are dominated by POD 2 and 1 respectively, and the mistuning parameter values for the other POD feature (POD 1 for sector 12 and POD 2 for sector 15)are very small, i.e. the mistuning values in sectors 12 and 15 are closely aligned to the directions of POD 2 and 1 respectively. Therefore the mode shapes (Φ_{a12} and Φ_{a15}) from cyclic analyses of sectors 12 and 15 can be used to form the bases for reduced-order model in Eq. (8) as they will be close to mode shapes (Φ_2 and Φ_1) in Eq. (3). Cyclic analyses are run for the two sectors 12 and 15 and mode shapes from the first five families are used in MMDA analysis.

Figure 5 plots deviations in frequencies estimated by MMDA and exact full rotor analysis (ANSYS) as given by Eq. (11) and (12), respectively. Figure 6 plots the % errors in estimation as given by Eq. (13).

$$Dev_{MMDA} = Freq_{MMDA} - Freq_{Tnd} \tag{11}$$

$$Dev_{Act} = Freq_{Act} - Freq_{Tnd} \tag{12}$$

$$ErrorD(\%) = \frac{Dev_{MMDA} - Dev_{Act}}{Dev_{Act}} * 100 \tag{13}$$

where $Freq_{Tnd}, Freq_{MMDA}, and\ Freq_{Act}$ are natural frequencies of the tuned disk, the mistuned disk via MMDA and the mistuned disk via full (360 degree) rotor analysis, respectively. As observed from these plots the estimates of natural frequencies from MMDA based on mode shapes from actual sectors as bases are very close to the actual values, which suggests that the use of alternative bases is valid for MMDA analysis. It should be noted that a few large values of $ErrorD(\%)$ are due to small values of corresponding Dev_{Act} in Eq. (13).

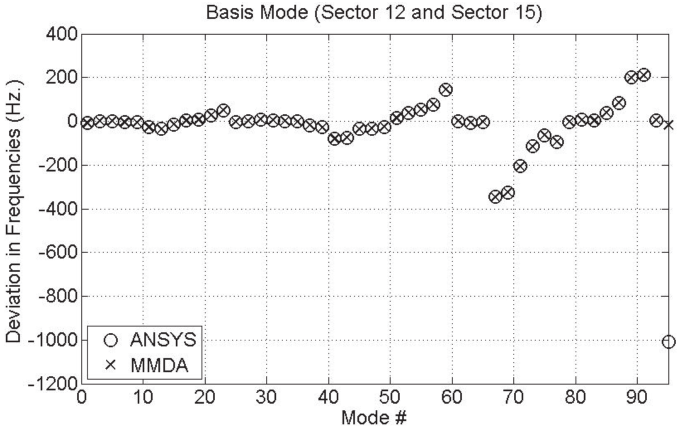

Figure 5: Deviations in natural frequencies estimated via full rotor FEM and MMDA (Alternative Bases, Sectors 12 and 15)

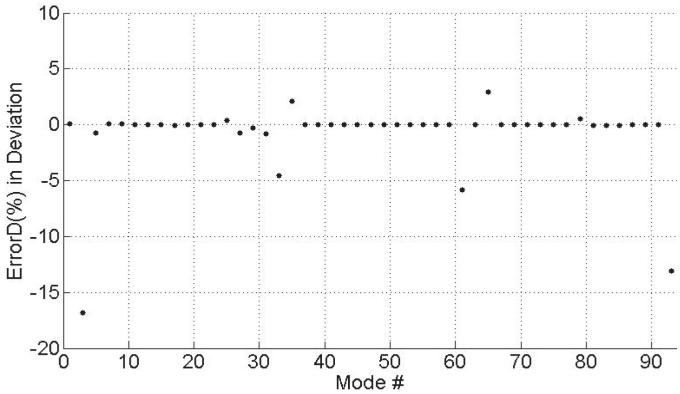

Figure 6: Errors (%) in deviations of natural frequencies estimated via MMDA (Alternative Bases, Sectors 12 and 15)

Since one of the main objectives behind reduced-order model is to use it in the harmonic analysis to estimate the amplitude magnification, it is essential that the reduced order model not only provides accurate estimates of the natural frequencies and mode shapes but also of the blade amplitudes under forced response or harmonic excitation. In order to verify the accuracy of the harmonic response, the system is excited by 3rd engine order excitation in ±3% range of the mean forcing frequency of 4157.6 Hz, which is the natural frequency of the nominal tuned disk for the 3 nodal diameter solution of the first family. The modal damping ratio is taken to be 0.1%. The harmonic responses obtained using modal superposition. For MMDA based analysis, the mode shapes from the analysis discussed earlier are used whereas for the full 360 degree rotor the mode shapes from ANSYS are used for modal superposition.

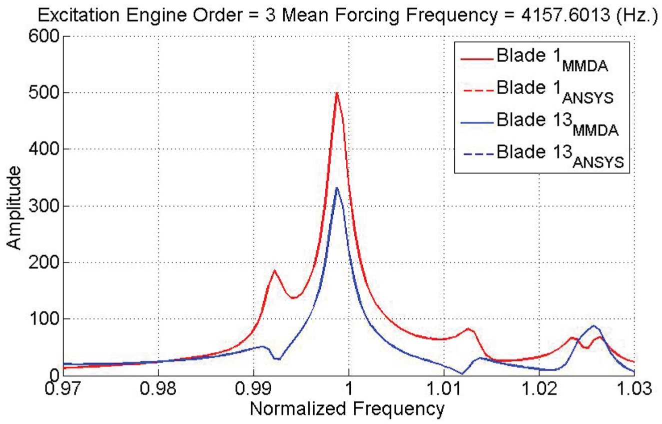

Figure 7: Blade tip amplitude estimated via full rotor ANSYS and MMDA (Alternative Basis, Sectors 12 and 15) for 3EO excitation

Figure 7 shows the blade tip amplitude for blade # 1 and 13. As observed from the figure, MMDA accurately estimates the harmonic response of the bladed disk assembly.

In a mistuned bladed disk assembly many POD features are present but only a few are dominant and have to be included in MMDA analysis. In order to simulate this, perturbation along an additional POD feature is introduced as shown in Figure 2. POD features 1 ($\mathbf{u}_1$) and 2 ($\mathbf{u}_2$) are same as the ones discussed earlier and are given by Eq. (14) and (15) respectively. POD feature 3 ($\mathbf{u}_3$) is created by taking the component of a specified vector $\mathbf{v}$ (Eq. 16) so that it is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$ via Gram–Schmidt ortho-normalization (Weisstein); i.e.,

$$\mathbf{u}_1 = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]^T \tag{14}$$

$$\mathbf{u}_2 = [-1 \quad -0.6 \quad -0.2 \quad 0.2 \quad 0.6 \quad 1]^T \tag{15}$$

$$\mathbf{v} = [1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1]^T \tag{16}$$

$$\mathbf{w} = \mathbf{v} - proj_{\mathbf{u}1}(\mathbf{v}) - proj_{\mathbf{u}2}(\mathbf{v}) \tag{17}$$

$$\mathbf{u}_3 = \frac{\mathbf{w}}{\|\mathbf{w}\|} \tag{18}$$

Here $\|\mathbf{w}\|$ ($= \sqrt{\mathbf{w}^T \mathbf{w}}$) is the norm of $\mathbf{w}$ and $proj_{\mathbf{u}_*}(\mathbf{v})$ is the projection of vector $\mathbf{v}$ on vector $\mathbf{u}_*$ and is given by:

$$proj_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{v}^T \mathbf{u}}{\mathbf{u}^T \mathbf{u}} \mathbf{u} \tag{19}$$

The mistuning values along POD features 1 and 2 are same as those taken earlier. Mistuning along POD feature 3 is intended to be minor, whose values are generated in Matlab using *randn* (MATLAB Help Manual, 2008) function and are given in Figure 8. The mean and standard deviation of the mistuning parameter (ξ_3) distribution are 0.0017 and 0.004 respectively. The standard deviation of ξ_3 suggests that it is about 25% of the dominant POD features, i.e. POD features 1 and 2.

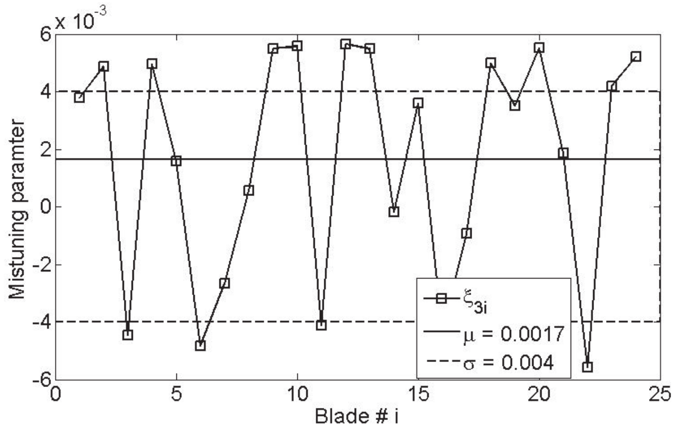

Figure 8: Mistuning pattern for POD #3

Due to the random nature of perturbations and hence the mistuning coefficients, it is possible that the blades most aligned in the direction of POD features may have a very small value of mistuning. In this case, the mode shapes obtained from the cyclic analyses of these blades will be very similar to the mode shapes of ‘average’ assembly and no additional information will be introduced by including these mode shapes in the solution bases. Hence an alternative approach for finding the blades most suitable for forming the bases is discussed. When selecting the blades the following conditions can be considered:

The selected blades should:

1. have perturbation close to the average perturbation of the blades’ geometries. This condition is required to prevent under or over stiffening of mode shapes in the bases.
2. represent independent directions of perturbations.

In order to satisfy condition 1, we can calculate the norm of perturbation $\delta\mathbf{p}_\ell$ (Eq. 20) for each blade and then choose the blade whose norm is closest to the median.

$$\|\delta\mathbf{p}_l\| = \sqrt{\delta\mathbf{p}_l^{\,T}\, \delta\mathbf{p}_l} \tag{20}$$

$\delta\mathbf{p}_l$: geometry perturbation vector for blade #*l*, $1 \le l \le n$

Once the first blade is selected, the other blades satisfying condition 2 can be successively selected by taking the projection of perturbation in the direction normal to the hyper-plane defined by the previously selected blades and selecting the blade with the maximum projection.

The algorithm discussed above is employed in selecting the blades for calculating the basis vectors for this example. The norm of perturbation for each blade is calculated and presented in Table A-1. The median of the norm of perturbation of the blades’ geometries is 0.0031. Blade 4 with a norm of 0.0031 is selected as the first blade. The second blade is selected by taking the projection of perturbation of blades (#1-3, 5-24) in the direction normal to the perturbation of blade#4 and choosing the blade with the maximum projection. The magnitudes of projections are presented in Table A-1 and blade #23 is selected as the second blade to form the basis. Once the blades for forming the basis have been selected, cyclic analyses are run for the two sectors and mode shapes from the first five families are used in MMDA analysis.

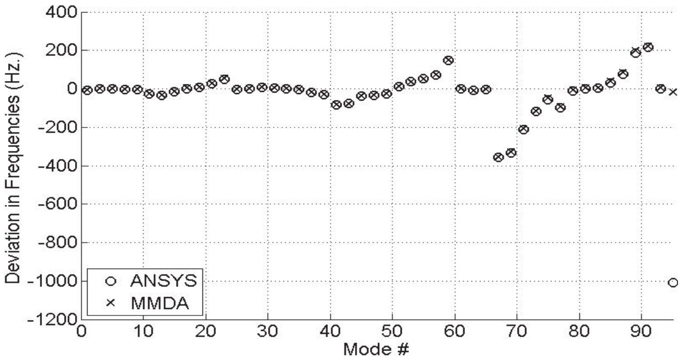

Figure 9: Deviations in natural frequencies estimated via full rotor ANSYS and MMDA (Alternative Bases, Sectors 4 and 23)

Figure 9 plots deviations in frequencies as estimated by MMDA and exact full rotor analysis as given by Eq. (11) – (13). As observed from the plots the estimates of natural frequency from MMDA based on mode shapes from actual sectors as bases are very close to the actual values.

As discussed earlier, since the final objective of reduced order analysis is to estimate the forced or the harmonic response of the blades, the accuracy of the results from the reduced order model is verified for the harmonic response as well. The system is excited under the same conditions as for the previous example, i.e. 3rd engine order excitation in ±3% range of the mean forcing frequency of 4157.6 Hz and the modal damping ratio of 0.1%. The harmonic response is obtained using modal superposition and the results are compared with the harmonic response from full rotor ANSYS analysis in Figure 10. As observed from the figure, the harmonic response estimated via MMDA is very close to the exact response from ANSYS analysis.

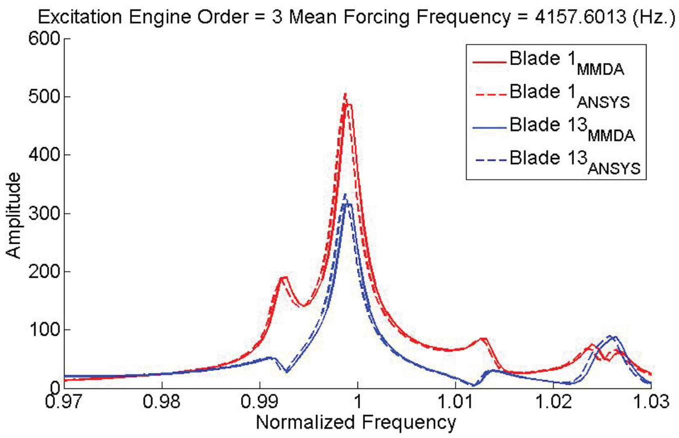

Figure 10: Blade tip amplitude estimated via full rotor ANSYS and MMDA (Alternative Basis, Sector 4 and 23), 3EO excitation

A comparison of Figures 7 and 10 shows that the harmonic response is very similar in nature, which is expected because, the mistuned bladed disks in the two examples only differ in mistuning along POD3 (bladed disk for Figure 7 has no mistuning along POD3), which is the minor POD feature and does not significantly alter the mode shapes or harmonic response. It is also observed that although the harmonic response estimate in Figure 10 is very close to the true response, it is not as accurate as the estimate in Figure 7; which is also expected because although minor, some mistuning is present along POD3 and only 2 dominant POD features, and hence 2 sectors are used to form the bases.

EXTREMELY LARGE MISTUNING

Another problem that arises in mistuning is that of the rogue blade, i.e. blade having geometry significantly different from the average geometry, which can be caused by foreign object damage (FOD) or blade tip blending to remove blade corrosion. In such cases the mode shape of the rogue blade is significantly different from the 'average' mode shape and the techniques discussed so far may fail to provide accurate results. But as it has been observed from the results of alternative bases, the mode shapes of actual blades can be used to form the bases. Same idea can be extended in case of extremely large mistuning and the mode shapes from the cyclic analysis of the sector with rogue blade, Φ_{rogue}, can also be included in bases to consider the impact of extremely large mistuning. Hence in the presence of extremely large mistuning the following modification to the bases in MMDA algorithm is suggested:

$$\Phi = [\Phi_0 \quad \Phi_1 \quad \Phi_2 \quad . \quad . \quad \Phi_{np} \quad \Phi_{rogue}] \quad (21)$$

Eq. (21) is similar to Eq. (3) where Φ_0 are tuned modes of the system with blades having the mean geometry and Φ_l are tuned modes of the system with blades having perturbed geometry along l^{th} POD feature (the rogue blade is not included in the POD analysis). Φ_{rogue} is composed of the tuned modes from the cyclic analysis of the sector with the rogue blade. The explicit inclusion of the mode shapes from the rogue blade in the bases is done to account for the large changes in the mode shapes of the rogue blade.

Numerical Results

MMDA with modified bases is applied to bladed disk with extremely large mistuning in this example and the results are presented. The bladed disk considered earlier in previous examples is considered again. Mistuning is applied along the POD #1, i.e. along the thicknesses of the blades. Mistuning parameter values along POD #1 are again given by Figure 3. Blade #23 is the rogue blade which has additional large mistuning along POD features #2 and #3, represented by Figure 2. The values of mistuning parameters for the rogue blade are given in Table 1.

Table 1 : Mistuning parameters for the rogue blade (Blade#23)

ξ_1	ξ_2	ξ_3
-0.027854	0.0667100	0.0608370

The mean value (excluding the rogue blade) of norms of perturbation vectors (Eq. 20) is 0.0025, whereas the norm of perturbation vector for the rogue blade is 0.0145, i.e. the perturbation in the rogue blade geometry is 5.8 times the average perturbation value.

MMDA analyses are run for the first family of the modes. In order to consider the impact of extremely large mistuning, first analysis is run without including the mode shapes of the sector with rogue blade, i.e. only the 'average' mode shapes and the mode shapes from geometry perturbed along POD feature 1 are included in the bases. Then the mode shapes from the rogue blade, Φ_{rogue}, are also included in the MMDA analysis. Full rotor ANSYS analysis is also run to compare the estimated natural frequencies with the true values. The deviations in frequencies and % errors in deviations, as given by Eq. (11), (12) and (13) are presented in Figures 11 and 12, respectively.

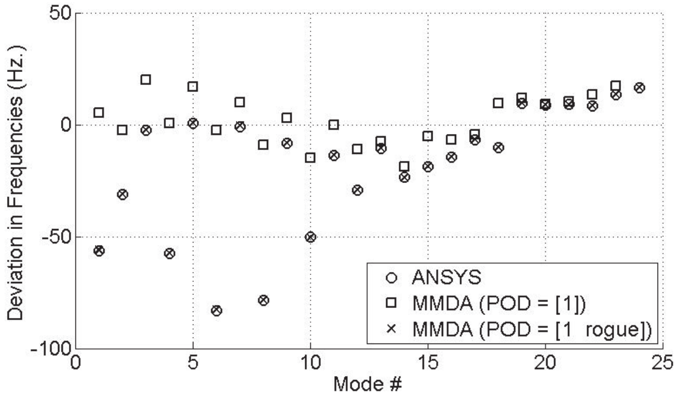

Figure 11: Deviations in natural frequencies estimated via full rotor ANSYS, MMDA (POD 1 only) and MMDA (POD 1 and rogue blade)

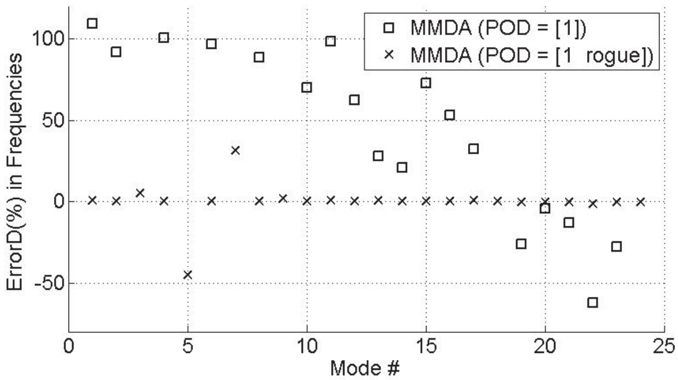

Figure 12: Errors (%) in deviations of natural frequencies estimated via MMDA (POD 1 only) and MMDA (POD 1 and rogue blade)

As discussed earlier, in the presence of extremely large mistuning, the inclusion of just the 'average' modes and the modes from POD analysis is not sufficient to form suitable bases, as observed from the results of MMDA analysis with only the 'average' and POD #1 modes in Figures 11 and 12. We can see large errors in the natural frequency estimates from the MMDA analysis without the rogue blade mode shapes. It can also be observed from Figures 11 and 12 that inclusion of mode shapes from the rogue blade rectifies this problem, and the natural frequency estimates from the MMDA analysis with rogue blade are very close to the true values.

In order to see the accuracy of harmonic response, the mistuned bladed disk is subjected to 3rd engine order harmonic excitation in ±3% range of the mean forcing frequency of 4157.6 Hz and with the modal damping ratio of 0.1%. The harmonic response is obtained using modal superposition and the results are compared with the harmonic response from full rotor ANSYS analysis in Figure 13.

As expected since the mode shapes from only the 'average' and POD analysis are not sufficient to form the suitable bases, the blade tip amplitude estimated from MMDA analysis without the rogue blade (Figure 13a) differ significantly from the true values. On the other hand the blade tip amplitudes estimated via MMDA with rogue blade match exactly with the true response. Hence, it can be said that with the explicit inclusion of mode shapes from the rogue blade in the basis, MMDA analysis can be applied even in case of extremely large mistuning.

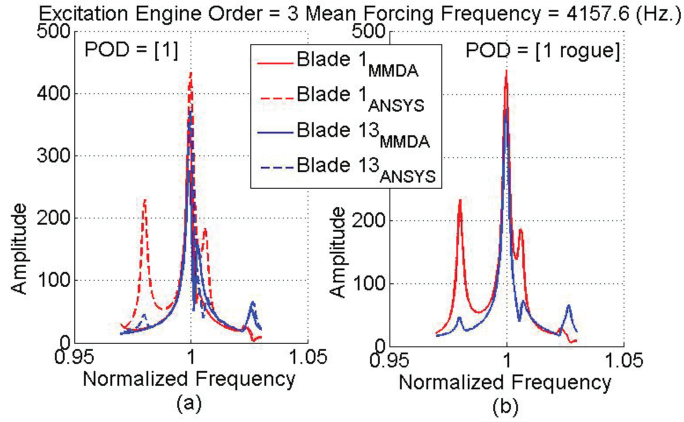

Figure 13: Blade tip amplitude estimated via full rotor ANSYS, MMDA (POD 1 only) and MMDA (POD 1 and rogue blade) for 3EO excitation

CONCLUSIONS

Modified Modal Domain Approach (MMDA) has been modified to use mode shapes from the cyclic analyses of the actual sectors to form the bases. The advantage of using the alternative bases is that this avoids the step of creating POD perturbed geometries to calculate the mode shapes in the bases. It has been shown that MMDA with alternative bases is able to accurately estimate the mistuned natural frequencies and harmonic response even when minor PODs are not considered in selection of blades to form the bases. Hence it preserves the accuracy and computational efficiency of the standard MMDA analysis, while reducing a step in the inputs generation for MMDA analysis.

It has also been shown that the idea behind alternative bases can be extended to cases of extremely large mistuning, and the mode shapes from the rogue blade (blade with extremely large mistuning) can be explicitly included in the basis vectors. The results from MMDA analysis on an academic rotor with extremely large mistuning in one of the blades show that although MMDA without the explicit inclusion of mode shapes from the rogue blade does not provide accurate results, the inclusion of mode shapes from the rogue blade rectifies this situation and the results from MMDA analysis with such modified bases are accurate.

It should be noted that result are presented for an academic rotor here. At present, MMDA is being applied to a real rotor with many POD features in blades' geometries variations.

ACKNOWLEDGEMENTS

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<u>References</u>

Bhartiya, Y. and Sinha, A., "Reduced Order Model of a Bladed Rotor with Geometric Mistuning: Comparison Between Modified Modal Domain Analysis and Frequency Mistuning Approach," GT2011-45391, presented at the *ASME Turbo Expo*, Vancouver, BC, Canada, June 6 – 11, 2011.

Bhartiya, Y. and Sinha, A., "Reduced Order Modeling of a Bladed Rotor with Geometric Mistuning via Estimated Deviations in Mass and Stiffness Matrices," *ASME Journal of Engineering for Gas Turbines and Power*, Vol. 135, No. 5, May 2013. (Also, DETC 2011-47495).

Brown, J. M., "Reduced Order Modeling Methods for Turbomachinery Design", Ph.D. dissertation, Wright State University, Dayton, Ohio, 2008.

Ganine, V., Legrand, M., Michalska, H., and Pierre, C. , " A

Sparse Preconditioned Iterative Method for Vibration Analysis of Geometrically Mistuned Bladed Disks," *Computers and Structures*, Vol. 87, 2009, pp. 342 – 354.

Ghanem, R.G., and Spanos, P.D., *Stochastic Finite Elements: A Spectral Approach*, Springer, New York, 1991

Lim, S., Castanier, M., and Pierre, C., "Vibration Modeling of Bladed Disks subjected to Geometric Mistuning and Design Changes," Proceedings of the 45th *AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference*, Palm Springs, CA, 2004, AIAA Paper No. 2004-1686.

MATLAB Help Manual, Twelfth Edition, The Mathworks Inc., Natick, MA, October, 2008.

Sinha, A., "Reduced-Order Model of a Bladed Rotor with Geometric Mistuning," *ASME Journal of Turbomachinery*, Vol. 131, 031007-1 to 031007-7, July 2009. (Also, ASME GT 2007 – 27276).

Sinha, A. and Bhartiya, Y., "Modeling Geometric Mistuning of a Bladed Rotor: Modified Modal Domain Analysis," *IUTAM Symposium on Emerging Trends in Rotor Dynamics, K. Gupta (ed.)*, IUTAM Book Series, Springer, 2010, pp. 177 – 184.

Sinha, A., Hall, B., Cassenti, B. and Hilbert, G., "Vibratory Parameters of Blades from Coordinate Measurement Machine Data," *ASME Journal of Turbomachinery*, Vol. 130, 011013-1 to 011013-8, January 2008. (Also, ASME GT 2005 – 69004).

Weisstein, E. W. "*Gram-Schmidt Orthonormalization.*" From MathWorld -- A Wolfram Web Resource. http://mathworld.wolfram.com/Gram-SchmidtOrthonormalization.html

Appendix

Table A-1: Norms of perturbation vectors and projection of perturbation normal to blade #4

Blade #	Perturbation Norm	Projection Norm
1	0.0026	0.0015
2	0.0023	0.0017
3	0.0055	0.0023
4	0.0031	-
5	0.0032	0.0020
6	0.0031	0.0029
7	0.0042	0.0007
8	0.0026	0.0012
9	0.0045	0.0026
10	0.0028	0.0027
11	0.0058	0.0044
12	0.0028	0.0028
13	0.0022	0.0010
14	0.0055	0.0047
15	0.0056	0.0029
16	0.0016	0.0016
17	0.0009	0.0007
18	0.0011	0.0007
19	0.0032	0.0015
20	0.0025	0.0019
21	0.0034	0.0013
22	0.0015	0.0014
23	0.0062	0.0051
24	0.0043	0.0006